PRIMUS
https://doi.org/10.1080/10511970.2021.1872750

Taylor & Francis
Taylor & Francis Group

Check for updates

Tangible Topology Through the Lens of Limits

Barbara A. Shipman and Elizabeth R. Stephenson

ABSTRACT
Point-set topology is among the most abstract branches of mathematics in that it lacks tangible notions of distance, length, magnitude, order, and size. There is no shape, no geometry, no algebra, and no direction. Everything we are used to visualizing is gone. In the teaching and learning of mathematics, this can present a conundrum. Yet, this very property makes point set topology perfect for teaching and learning abstract mathematical concepts. It clears our minds of preconceived intuitions and expectations and forces us to think in new and creative ways. In this paper, we present guided investigations into topology through questions and thinking strategies that open up fascinating problems. They are intended for faculty who already teach or are thinking about teaching a class in topology or abstract mathematical reasoning for undergraduates. They can be used to build simple to challenging projects in topology, proofs, honors programs, and research experiences.

KEYWORDS
Topology; teaching; convergence; sequences

1. INTRODUCTION

A striking bumper sticker in Norway reads "I ◯ topology." This absurd statement captures the essence and fascination of the unusual realm of topology, the mathematical goggles through which we see worlds without corners or curvature, without length, smoothness, order, or magnitude. In topology, ♡, ◯, △, and □ are all the same, as are various letters of the alphabet [6], or a bagel and a coffee mug.

The one feature that topology cares about in the shape ♡ is the hole in the middle. Knowing where the holes are is an interesting question, and a practical one too. Seemingly unrelated data sets, such as those arising from music and electromagnetic waves, can be analyzed using similar topological methods [10, 11]. Other familiar components of modern life also reduce to topology, such as making schematic maps of train stations, coloring maps, and untangling tangled wires. These allow for interesting hands-on activities that fit nicely into a school curriculum, as outlined, for example, in [14].

But how about point-set topology? Here, no geometry or algebra is left at all, not even holes. All we have is a set of points and a simple recipe for creating subsets from it. In a world where simple, everyday observations such as corners, shape, angle, and

CONTACT Elizabeth R. Stephenson elizabeth.stephenson@ist.ac.at Institute of Science and Technology Austria, Klosterneuburg 3400, Austria.

© 2021 Taylor & Francis Group, LLC

now even holes, no longer matter, point-set topology leaves us estranged from our familiar and concrete senses. As faculty, we have developed abstract mental images for working with these ideas. But how can we internalize or communicate something so seemingly intangible to our students? And why should they be interested in such things?

For the student still clinging to the presumption that the only important areas of mathematics are those that directly impact the real world, we note that even this incredibly abstract discipline climbs down from its ivory tower and interacts with us from time to time. For example, it proved quite useful to define spatial relations independent of a distance function for use in geographic information systems [3].

But the most important impact point-set topology has is that it forces us to think about mathematics in new and creative ways. It clears our minds of preconceived intuitions and expectations. It shows us that things are not always as our experience and schooling tell us they ought to be. Our minds are challenged to look for the impossible. And in this foreign, uncharted territory, what we discover is astonishing.

Out of our experiences in taking and in teaching topology, proofs, and undergraduate research projects, we have created guided discoveries for faculty seeking to stimulate advanced, abstract, and creative thinking in undergraduate courses on topology and mathematical reasoning.

The questions are accompanied by goals in effective thinking that the investigations are designed to develop and strengthen. We recommend the book "The 5 Elements of Effective Thinking" [2] by two renowned teachers of mathematics as a valuable companion to this paper.

For one of the authors, these investigations have been tremendously beneficial. During an interview for admission to graduate work in mathematics, she felt underprepared to give insightful answers to questions from standard coursework, but when asked about her experiences in topology, based on the discoveries presented in this manuscript, she impressed the interviewer with the depth of her understanding. Long hours of pondering the questions formulated here proved useful indeed.

The directions of thought that we take in these discoveries and the questions they lead to have a decisively different flavor from typical textbooks on topology or proofs. The flavor is investigative, suspenseful, and captivating. Though the questions we ask are fundamental and easily stated, we have not found them in any book or resource on topology. To one of them (Question 7), we still do not have an answer, despite its simple statement!

2. THE BIRTH OF TOPOLOGY

Before jumping into the strange and unknown world of point-set topology, it is helpful for students to see where topology arises historically in mathematics. Although more fundamental in nature, topology became a full-fledged field of mathematics later than its classical counterparts of analysis, differential geometry, and modern algebra, which look at more visibly tangible structures.

Topology made a famous appearance in 1736, when Euler published a paper entitled *Solutio problematis ad geometriam situs pertinentis* [4] (*The solution of a problem relating to the geometry of position*). In this work, Euler proposed a solution to the Königsberg Bridge problem in which he removed from consideration all apparent features of the problem that were not inherently relevant to the question. We like to make this simple, but extremely effective, mindset explicit to our students:

Thinking Strategy 1 (getting to the bottom): What is the real, underlying question? Strip away everything not relevant to it.

In this famous problem, the city of Königsberg encompassed the Pregel River and two large islands, which were linked to the mainlands by seven bridges. The task was to devise a path through the city crossing every bridge exactly once. Euler, realizing that this apparently geometrical question did not rely on geometry at all, cleverly proved that the problem has no solution. In doing so, he reduced the problem and its outcome to topology, where analytic and geometric features are irrelevant.

Seeing this strategy of thought in the historical work of Euler highlights its importance and power. It is insightful for students to look for where this principle can simplify and clarify their own thoughts.

Question 1 (applying Euler's strategy): How can Euler's strategy of getting to the bottom of the question help you clarify other problems or ideas you are trying to understand?

When students feel confused, the simple task of figuring out exactly what the question is may help them to see a solution. This strategy may seem obvious, but it is deceptively difficult to apply. With practice, however, it eventually becomes habit. This habit was monumental in the historic work of Andrew Wiles in his proof of Fermat's Last Theorem in 1992. After an error was discovered in his 200-page proof and months of attempts to fix it failed, Prof. Wiles sat down to settle the question of just why it was not working. Then, in a revelation that he described as sudden and totally unexpected, he saw what he needed to correct the proof – a very simple "3, 5 switch." The captivating story is made accessible to a general audience in a documentary, the transcript of which may be found in [7].

It is noteworthy that the field of topology was born out of this "minimalist" mode of thinking. Through the centuries that followed Euler's solution to the Königsberg bridge problem, mathematicians added to Euler's foundation bit by bit, many inspired by ideas from analysis. Cantor, Riesz, Fréchet, Hausdorff, Bernoulli, Poincaré, and Schmidt were especially instrumental in advancing the field, and their contributions, like Euler's, were specific solutions to individual problems rather than systematic studies of the new mathematics itself. Finally, in 1912, Brouwer amassed the disparate parts into a uniform whole, and the field of formalized topology was born. The mathematics that arose now had the power to move and breathe on its own, bringing with it new questions and mysteries [8].

3. MENTAL IMAGERY

Point-set topology is now its own discipline, separate from the concrete, everyday world we are familiar with, but rich in beautifully expressed logic. Because of its strange, unnatural appearance, many people – even mathematicians – have only a passing appreciation for this lovely field of mathematics.

How can we understand topology? A common expression of understanding is to say, “I see!” But when people talk about seeing mathematics, they often mean something more concretely related to sight, such as the beautifully intricate pictures of Julia sets. While computer graphics have led to better insight into the world of mathematics [5, 9], mathematical concepts such as Dirichlet’s function may be technically impossible to draw. As mathematics becomes increasingly abstract, accurate pictorial representation of it becomes futile. The mathematician must rely on a well-developed ability to visualize foreign ideas, not with the eyes or with memories of things previously seen, but with creative imagination and rigorous intuition [1].

Mental images should be true to the mathematical meaning and versatile for the mind to play with. There may be aspects of the image that can be communicated verbally or on paper, but in its entirety, the image lies in the mind. One of the authors, in her undergraduate work, understood the definitions of convergence in analysis more quickly than some of her classmates because of the mental visualization from her prior experience with topology, in particular the notions of actual versus potential infinity [12]. When she shared these mental images with her friends, they understood definitions on convergence more easily as well.

4. CONCEPTS OF CLOSENESS

Since the complexities of point-set topology flow from the simple, abstract definition of a topology, we want our students to see where the definition comes from. So we bring them back to their study of calculus, where notions of closeness guide our understanding.

Students will have encountered questions such as these: How close are the terms of a sequence to the proposed limit? How close is the slope of a secant line to the slope of the tangent? How close are the upper and lower sums in approximating an integral? A sense of distance is so embedded in our concept of a real number that we name the number by how close it is to zero!

More abstractly, we think of closeness to a point p by looking at a radius around p. For example, the points at a distance less than 5 from 0 make an open interval on the real line (or an open disk in the plane, or an open ball in 3-space). On the real line, open sets are unions of open intervals. One can check that this is equivalent to

Definition 1 (open sets of real numbers): To say that a set of real numbers S is *open* means that for every $p \in S$, there is an open interval around p that is completely contained in S.

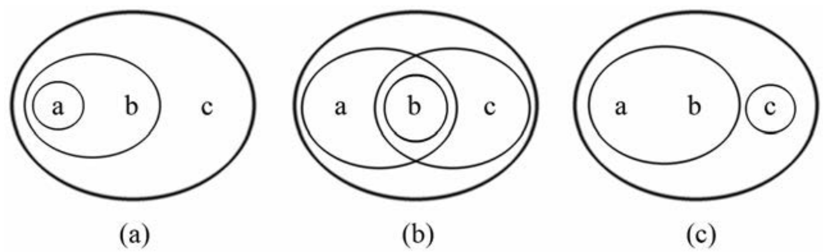


Figure 1. Three topologies on $X = \{a, b, c\}$. In each topology, the nonempty open sets are circled.

In topology, where there is no concept of distance, and so no concept of interval, open sets will be used to test for closeness. To help students determine what "open" should mean more abstractly, we first ask them to look at basic properties of open sets on the real line.

Question 2 (properties of open sets in $\mathbb{R}$): (a) Is the whole real line open? (b) Is the empty set open? (c) If we take any union of open sets in $\mathbb{R}$, is the result still open? (d) If we take the intersection of finitely many open sets in $\mathbb{R}$, is the result still open? (e) If we take any intersection of open sets in $\mathbb{R}$, is the result still open?

Students will find that to the first four questions, the answer is "yes." For the last one, they can look for a counter-example, such as the intersection of the sets $(-1/n, 1/n)$ as n runs over the counting numbers.

We now have four simple properties of open sets in the real line. We will use them to create a topology on any set of points! To highlight the process for the students, we make the thinking strategy explicit:

Thinking Strategy 2 (abstraction from the concrete): Create abstract concepts by extracting properties of familiar things.

Now take the (true) properties of open sets in $\mathbb{R}$ that we found in Question 2 and transfer them to a collection of subsets from an arbitrary set of points. This gives us the definition of a point-set topology:

Definition 2 (a topology): A *topology* on a set X is a collection of subsets of X, called *open sets*, such that (1) X and the empty set are open, (2) arbitrary unions of open sets are open, and (3) finite intersections of open sets are open. The set X, together with its topology, is called a *topological space*.

The rules are simple. Begin with any set X that you wish, and choose a collection of subsets of X so that the three properties hold. This is a topology on X. Nothing else is needed. Point-set topology and all of its intricacies rest on this definition alone.

To help students understand this definition, we invite them to start simple, say, with a set of three points. A few topologies they will discover are shown in Figure 1.

Thinking Strategy 3 (start simple): Look for simple examples of abstract things.

Question 3 (simple topologies): How many topologies are there on a set of three points? What topologies are there on a set of two points? four points? How about one point? Can the set X be empty?

5. TOPOLOGIES ON THE REAL NUMBERS

To build a topology on the set of real numbers, all we really need is a set of objects that has the same cardinality as the real numbers. But the real numbers are convenient because they already have names! Here students can appreciate how Thinking Strategy 1 comes into play again. We strip away everything about the real numbers except the points and their names. What the elements are, or what properties they have, do not matter. It is as if we took a line of marbles and dumped it into a bag and shook it all up. Names such as "5" and "π" are now only names; they convey no magnitude or ordering or rational or irrational character. The real "line" is no longer a line. It is simply a set.

We may now create topologies on $\mathbb{R}$ that are completely out of line with our usual thinking, and with a completely different flavor from the standard Euclidean topology. All we need is a collection of open sets that satisfies the three properties in Definition 2. How shall we begin?

Thinking Strategy 4 (understand the extremes): Look for extreme cases of abstract things to explore the limits of what is possible.

Question 4 (smallest and largest topologies): (a) Can a topology on $\mathbb{R}$ have no open sets? If not, what are the fewest open sets it can have? (b) If we let every subset of $\mathbb{R}$ be open, does this make a topology?

Students will discover that at the two extremes are the trivial topology, where only $\mathbb{R}$ and the empty set are open, and the discrete topology, where every subset of $\mathbb{R}$ is open. Note, for example, that in the discrete toplogy, $[0, 1]$ is open, while in the trivial topology, $(0, 1)$ is not open! This will put students on alert to expect strange things.

Let us look more closely at the discrete topology on $\mathbb{R}$. The familiar sequence $(1/n)$ may help students feel at home. Surely they will recall that in the world of calculus, $(1/n)$ has a unique limit and that limit is 0! We nudge them out of their comfort zone with a simple question.

Question 5 (a simple exploratory question): In the discrete topology on $\mathbb{R}$, does the sequence $(1/n)$ converge to 0 and only to 0, as usual? How does the topology help us to check?

It is fun to stop here and let the students think. Let them come to the realization themselves that the things they relied on for intuition in the standard topology are gone – there is no notion of distance to measure closeness to zero. (Remind them that $\mathbb{R}$ is just a set, and "1/5" is only a name!) They may want to use the definition

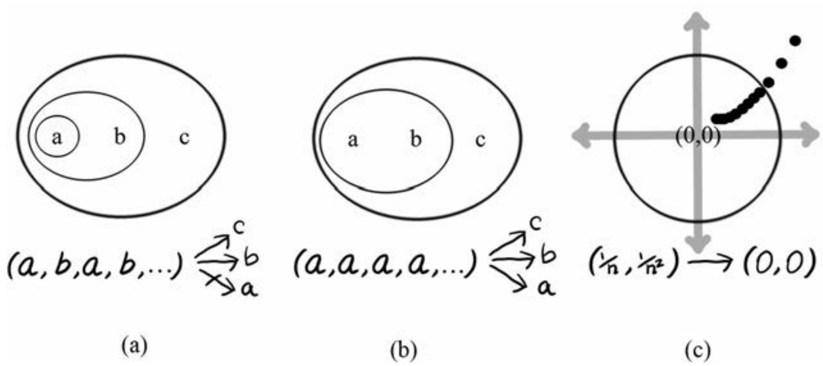


Figure 2. (a) A topology where $(a, b, a, b, \ldots)$ converges to b and c but not to a; (b) A topology where $(a, a, a, \ldots)$ converges to a, b, and c; (c) In the Euclidean topology on $\mathbb{R}^2$, the sequence $p_n = (1/n, 1/n^2)$ converges to $(0, 0)$.

of the limit of a sequence of real numbers. So we let them write it down. But alas, their definition most likely refers to a distance "epsilon" and wants us to consider the "distance from 0" of the elements of our sequence. But these notions do not exist in topology. We can't talk about them.

Students will realize that to check whether 0 is a limit of $(1/n)$ in this new scenario, we need to define what "limit" means in topology. First, let's re-examine the traditional meaning of limit, expressed in a less traditional way:

Definition 3 (limit of a sequence in $\mathbb{R}$ – usual meaning): Let (x_k) be a sequence in $\mathbb{R}$, and let $p \in \mathbb{R}$. To say that *p is a limit of* (x_k) means that every open interval $(p - r, p + r)$, with $r > 0$, contains all but finitely many terms of (x_k).

This is the definition expressed without reference to the natural numbers that index the terms [12]. It looks only at the radius r and how many terms lie within the given radius. It is helpful to pause here and let the students check that this definition is equivalent to the standard "ϵ, N" definition seen in their textbooks.

Notice that in Definition 3, all the concepts that are referred to, except for an "open interval," exist in a topology. But in a topology, we do have open sets. So we use Thinking Strategy 2 again, replacing open intervals containing p by open sets containing p:

Definition 4 (limit of a sequence in topology): Let (x_k) be a sequence in a topological space X, and let $p \in X$. To say that *p is a limit of* (x_k) means that every open set containing p contains all but finitely many terms of (x_k).

Using alternate language, we may refer to an open set containing p as a *neighborhood* of p. That is, instead of checking how many terms of (x_k) lie in arbitrary balls "close around p," we check how many terms live in arbitrary neighborhoods of p. Figure 2 illustrates some examples.

Students can now use Definition 4 to see if 0 is a limit of the sequence $(1/n)$ in the discrete topology. In the discrete topology, every subset of $\mathbb{R}$ is open. So take any subset of $\mathbb{R}$ containing 0. Does it contain all terms of $(1/n)$ except for finitely many? While some neighborhoods of 0 do, such as $[0, 5]$, others do not. For example, neither $[-1, 0]$ nor $\{0\}$ contains any term of $(1/n)$. So zero is not a limit of $(1/n)$! Students will see too that by the same reasoning, $(1/n)$ does not have any limit. One now wonders,

Question 6 (limits in the discrete topology): Is there any sequence that has a limit in the discrete topology? If so, what sequences do?

To answer Question 6, a new thinking strategy will be helpful:

Thinking Strategy 5 (satisfy a hardest condition): Among the conditions to be satisfied in a theorem or definition, is there one that is most difficult? If there is, try to satisfy it first.

To apply Thinking Strategy 5 to Question 6, we consider how hard it is to satisfy Definition 4 in the discrete topology. We look at every neighborhood of 0 and hope it contains all terms of the sequence except for finitely many. Larger neighborhoods make this easier and smaller neighborhoods make it harder. Does 0 have a smallest neighborhood in the discrete topology? If so, what is it?

Students will find that this neighborhood is $\{0\}$. How can a sequence have all of its terms in $\{0\}$, except for finitely many? It does if and only if all of its terms are 0, except for finitely many! That is, the sequence is eventually constant at 0.

Students will discover too that if $\{0\}$ contains all but finitely many terms of a sequence, then so does every other neighborhood of 0, and Definition 4 is satisfied. Thus we discover the answer to Question 6: In the discrete topology, a sequence (x_k) converges to p if and only if (x_k) is eventually constant at p. And, since a sequence can be eventually constant at only one number, this limit is unique.

The considerations above lead to further observations and questions that may be too advanced for an undergraduate course but will be interesting to consider as part of a research project. We devote the remainder of the section to these more challenging questions.

Eventually constant sequences converge under the most difficult conditions. If this is the only way a limit can occur, we say that the topology has the scarce limit property. We define it as follows:

Definition 5 (scarce limit property): To say that a topological space X has the *scarce limit property* (or is a *scarce limit topology*) means that if a sequence in X has a limit, then the sequence is eventually constant.

In a scarce limit topology, are limits unique? Suppose (a_n) is eventually constant at p but converges also to $q \neq p$. Then every neighborhood of q contains p. So the sequence $(p, q, p, q, \ldots)$ converges to q, and this contradicts the scarce limit property. So the limit is unique.

Another example of a scarce limit topology is the co-countable topology, where the open sets are $\emptyset$, $\mathbb{R}$, and every set whose complement is countable (that is, finite or countably infinite). May we reduce the topology further and still retain the scarce limit property? A far as we know, this is an open question.

Question 7 (scarce limit topologies): Does every scarce limit topology on a set X contain the co-countable topology on X?

6. TOPOLOGIES ON THE COUNTING NUMBERS

Here we juxtapose pairs of topologies on the counting numbers, $\mathbb{N}$, for students to consider that have contrasting limit behaviors. In doing do, we discover a property of a topology that makes it easy to test for limits.

One such pair of topologies are what we call the *stalagmite* and *stalactite* topologies. In the stalagmite topology, the open sets are $\mathbb{N}$, $\emptyset$, and the sets $\mathbb{N}_k = \{1, 2, \ldots, k\}$. One may imagine $\mathbb{N}_k$ as a stalagmite, extending upward from the floor of a cave, with highest point k. In the *stalactite topology*, the open sets are $\mathbb{N}$, $\emptyset$, and the sets $\mathbb{N}_k = \{k, k+1, k+2, \ldots\}$. $\mathbb{N}_k$ may be seen as a stalactite, reaching downward with lowest point k.

Question 8 (contrasting limit behaviors): In the stalagmite and stalactite topologies, what sequences converge to 4? For $p \in \mathbb{N}$, what sequences converge to p? How do the outcomes contrast?

Thinking Strategy 5 is helpful again. Is there a neighborhood of 4 in the stalagmite topology that is the smallest – the most difficult for a sequence to stay inside of? Students will find that there is – the neighborhood $\{1, 2, 3, 4\}$. Any sequence (x_n) with limit 4 must have all of its terms in $\{1, 2, 3, 4\}$, except for finitely many. That is, eventually, $x_n \leq 4$.

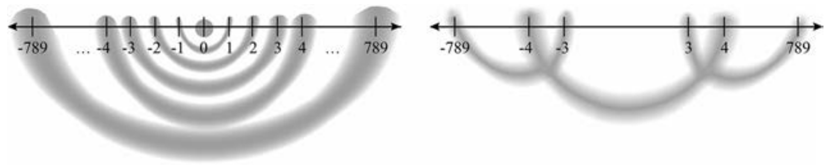
In the stalactite topology, the reverse is true: 4 has a "smallest" neighborhood, $\{4, 5, \ldots\}$. Any sequence with limit 4 must have all of its terms in $\{4, 5, \ldots\}$, except for finitely many. That is, eventually, $x_n \geq 4$. Students can now generalize these outcomes to any $p \in \mathbb{N}$.

What makes it easy to test for limits in these two topologies is that every point has a "minimal" neighborhood – the intersection of all of its neighborhoods. To see if a sequence satisfies Definition 4, we need only test this one neighborhood.

This strategy works in any topology where every point has a minimal neighborhood. We can therefore use Thinking Strategy 2 to see this property more generally in other topologies:

Definition 6 (minimal neighborhoods): Let p be an element of a topological space X, and suppose that the intersection of all open sets containing p is open. We call this intersection the *minimal neighborhood* of p. To say that X has the *minimal neighborhood property* means that every point in X has a minimal neighborhood.

As we have seen, minimal neighborhoods, when they exist, make it easy to test for limits. To check that a sequence (x_n) has limit p, we need only verify that the minimal neighborhood of p contains all terms of (x_n) except for finitely many.



(a) (b)

Figure 3. (a) The branches of the candelabra depict the smallest open sets in the candelabra topology: $\{0\}$ and the symmetric pairs $\{-n, n\}$; (b) Unions of the open sets in (a) create the other open sets of the topology, such as $\{-789, -4, -3, 3, 4, 789\}$.

Another pair of topologies with contrasting limit behaviors are the *single 5 topology* $\mathcal{T}_5$, with exactly three open sets, $\mathbb{N}$, $\emptyset$, and $\{5\}$, and the *excluded 5 topology* $\mathcal{T}_{\hat{5}}$, where the open sets are $\mathbb{N}$ and all subsets that exclude 5. Students should check that both of these collections of open sets satisfy Definition 2 of a topology.

We note that the excluded 5 topology is a specific example of the excluded point topologies on arbitrary sets studied in many books on topology, for example [13]. There are also the particular point topologies, where the open sets are $\emptyset$ and all sets that contain the chosen point.

Question 9 (minimal neighborhoods and limits): Does the single 5 topology have the minimal neighborhood property? What about the excluded 5 topology? What sequences have limit 5 in each of these topologies? Which have limit 7, or limit $p \neq 5$?

In playing with these topologies, students will see that both of them have the minimal neighborhood property. The minimal neighborhood of 5 is $\{5\}$ in $\mathcal{T}_5$ and $\mathbb{N}$ in $\mathcal{T}_{\hat{5}}$. So in $\mathcal{T}_5$, the sequences with limit 5 are eventually constant at 5, while in $\mathcal{T}_{\hat{5}}$, every sequence has limit 5. For 7, the situation is reversed: the minimal neighborhood of 7 in $\mathcal{T}_5$ is $\mathbb{N}$, and in $\mathcal{T}_{\hat{5}}$ it is $\{7\}$. Again, by testing only the minimal neighborhood, students will find that in $\mathcal{T}_5$, 7 is a limit of every sequence, while in $\mathcal{T}_{\hat{5}}$, the sequences with limit 7 are eventually constant at 7. They can then generalize this to any limit $p \neq 5$.

The foregoing four topologies on $\mathbb{N}$ may be adapted to finite and uncountable sets with similar contrasting outcomes.

Another topology, one that is most visually appealing in the set of all integers $\mathbb{Z}$, is the *candelabra topology*. Its open sets are arbitrary unions of the sets $\{-n, n\}$ and $\{0\}$, as shown in Figure 3(a); another open set is depicted in Figure 3(b).

Notice that in the candelabra topology, both 789 and -789 are limits of the constant sequence (789). In fact, any sequence with limit p also has limit $-p$. What makes this happen is that every open set containing p also contains $-p$, so that p and $-p$ behave as one element with two different names. If we identify p and $-p$ as one, $\mathbb{Z}$ becomes $\mathbb{N} \cup \{0\}$, the set of non-negative integers, and the candelabra topology becomes the discrete topology on $\mathbb{N} \cup \{0\}$. In essence, the candelabra topology is a double image of the discrete topology, like a view of a mountain and its reflection in a lake.

All five topologies featured in this section have the minimal neighborhood property. Are there topologies on $\mathbb{N}$ that do not? We leave this for the readers and their students to explore.

Question 10 (lack of minimal neighborhoods): Are there topologies on $\mathbb{N}$ for which no point has a minimal neighborhood, or where some, but not all points have minimal neighborhoods? What about topologies on $\mathbb{R}$?

7. TOPOLOGIES ON FINITE SETS

In the familiar world of applied mathematics, an oscillating sequence such as $(p, q, p, q, \ldots)$ with $p \neq q$ does not have a limit. But in topologies we have already seen, oscillating sequences can have limits, and perhaps many!

Our explorations will create interesting dissonance for the intuition and may be more challenging for students. They can be used as more advanced projects for teamwork or research experiences. We begin in topologies that are already familiar.

Question 11 (limits of an oscillating sequence): Does the sequence $(3, 4, 3, 4, \ldots)$ have a limit in the stalagmite or stalactite topologies, or in the single 5 or excluded 5 topologies? If so, what are the limits?

From Definition 4 and the minimal neighborhood property (Definition 6), students will find that in the stalagmite topology, any $L \geq 4$ is a limit of $(3, 4, 3, 4, \ldots)$. In the stalactite topology, the limits are 1, 2, and 3; in the single 5 topology, the limits are any $L \neq 5$, and in the excluded 5 topology, the unique limit is 5.

The most striking of these is perhaps the excluded 5 topology, where neither p nor q is a limit of $(p, q, p, q, \ldots)$ but something else is. To focus in on this strange behavior, we turn to Thinking Strategy 3 and ask,

Question 12 (looking for a simplest example): Is there a simplest topology in which neither p nor q is a limit of $(p, q, p, q, \ldots)$ but something else is?

Let us try to create such a scenario. Our set X must have at least two distinct elements, say 0 and $-$. Since we want $(0, -, 0, -, \ldots)$ to converge to something other than 0 or $-$, we need another element, say V. Now $X = \{0, -, V\}$. Since V is a limit of $(0, -, 0, -, \ldots)$, every neighborhood of V must contain both 0 and $-$. The only such neighborhood is the whole set X. Now to prevent $(0, -, 0, -, \ldots)$ from having 0 as a limit, 0 must be in an open set that does not contain $-$. We know already that V cannot be in it, so this leaves $\{0\}$ open. Similarly, $\{-\}$ is open. Students can check that the five open sets X, $\emptyset$, $\{0\}$, $\{-\}$, and $\{0, -\}$ make a topology on X. We call this the *winking owl topology*. It is illustrated in Figure 4(a). Strange as it may seem, the sequence $(0, -, 0, -, \ldots)$ has the unique limit V.

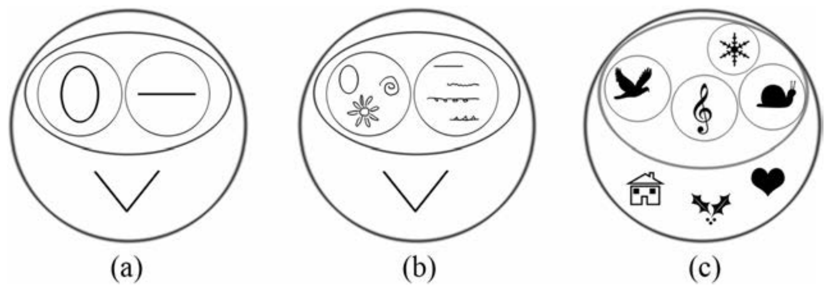


Figure 4. (a) The winking owl topology; (b) An expanded eye topology; (c) A parliament of owls topology (where just the four elements of the discrete part are circled). (b) and (c) are generalizations of (a).

Notice that in the winking owl topology, any sequence in which both 0 and $-$ occur infinitely many times has V as its unique limit. Soon we will see this as a specific example of what we call "digressive" convergence, since it converges to something other than the terms it is composed of.

Thinking back to the excluded 5 topology from the previous section, students will see a similarity to the winking owl topology as they consider the next question:

Question 13 (a generalized winking owl topology): How is the excluded 5 topology a generalization of the winking owl topology? Hint: picture the winking owl topology as an "excluded beak topology" where we think of the V as 5.

Using the hint in Question 13, and re-naming 0 as 2 and $-$ as 3, for example, students will see that the winking owl topology is a finite version of the excluded 5 topology. If we then extend the two "eyes" to all the counting numbers except for 5, we obtain the excluded 5 topology.

Question 13 calls for a thinking strategy that will help us discover other new and interesting topologies.

Thinking Strategy 6 (generalize a simplest case): When we find a simple example with an interesting property, consider how it may be viewed as a special case of more complex structures.

To generalize the winking owl topology in a different way, suppose we expand each of the eyes, $\{0\}$ and $\{-\}$, to a larger, but still finite set and leave everything else the same, as in Figure 4(b). We call these *expanded eye topologies*.

How does the convergence of $(0, -, 0, -, \ldots)$ to the unique limit V in the winking owl topology generalize to expanded eye topologies? That is,

Question 14 (limits in expanded eye topologies): In an expanded eye topology, what sequences have V as a unique limit but have no terms equal to V?

A good place for students to start is in writing down a few simple expanded eye topologies, perhaps one where each eye becomes a set of just two elements. Say we take $X = \{0, \oplus, -, +, V\}$, where the open sets are X, $\emptyset$, $\{0, \oplus\}$, $\{-, +\}$, and

$\{0, \oplus, -, +\}$. Students will find that as long as there is at least one element from each "eye" that occurs infinitely many times in the sequence, the sequence will converge only to V. Some examples are $(\oplus, +, +, \oplus, +, +, \ldots)$, $(\oplus, +, 0, \oplus, +, 0, \ldots)$, and any sequence of infinitely many 0's and infinitely many $-$'s.

To abstract these interesting limit behaviors further, we notice from Definition 4, that to find the limit (or limits) of a sequence with a finite range, we care only about the values that occur infinitely many times in the sequence. And because X is finite in this section, any sequence in X will have a finite range.

These observations lead to a generalization of the type of convergence we discovered in the winking owl topology, where a sequence with finite range has a limit that is different from any term that appears in the sequence. Our next definition captures this more generally for a sequence with finite range in any topology.

Definition 7 (digressive convergence): Let (x_n) be a sequence with finite range in a topological space X. The *base set* of (x_n) is the set of all elements of X that occur infinitely many times in the sequence. We say that two sequences are *equivalent* if they have the same base set. A sequence that converges to nothing in its base set but to at least one other element of X is said to have *digressive convergence*.

To help students grasp this more general definition, it is helpful to start with the simplest example, the winking owl topology.

Question 15 (digressive convergence in the simplest example): How many equivalence classes of sequences does the winking owl topology have? Which of them have sequences with digressive convergence?

In the winking owl topology, the base sets are few enough for students to write down; they are the seven non-empty subsets of $X = \{0, -, V\}$. There are seven equivalence classes of sequences, corresponding to these seven base sets. In the three classes where the base set is a single element, the sequences are eventually constant; they converge to the base constant and to V (these are the same when the base set is $\{V\}$). The sequences with base sets $\{0, V\}$, $\{-, V\}$, $\{0, -, V\}$, and $\{0, -\}$ have unique limit V; the convergence is digressive only for those with base set $\{0, -\}$.

After answering Question 15, students can consider it again, for expanded eye topologies. It turns out that any sequence whose base set contains at least one element from each expanded eye converges to V, and this limit is unique. The sequences with digressive convergence are those whose base set contains at least one element from each expanded eye and does not contain V.

To extend the winking owl topology in another way, replace the set $\{0, -\}$ by any finite set $\{e_1, \ldots, e_k\}$ with the discrete topology and augment V to a finite set $\{V_1, \ldots V_l\}$. We may think of this as an arbitrary number of eyes $e_1, \ldots, e_k$, each of which belongs to a singleton open set, and multiple beaks $V_1, \ldots V_l$ outside of this discrete sub-topology. These we call *parliament of owls* topologies. One rendition is shown in Figure 4(c). To create a sequence with digressive convergence in this topology, let the base set be any subset of $\{e_1, \ldots, e_k\}$ with two or more elements. Any such sequence converges to every V_j but to nothing in the base set.

We conclude with a question for readers and their students to contemplate.

Question 16 (topologies with digressive convergence): What other topologies on a finite set have sequences with digressive convergence? What about topologies on an infinite set?

8. REFLECTIONS

Through the lens of limits, we have discovered fascinating properties of topologies and captivating questions that, although elementary in nature, we have not seen presented in any textbook or discussion on topology: the scarce limit property, the minimal neighborhood property, and digressive convergence.

These investigations open up many more intriguing questions, outcomes, and directions of thought too long to fit in these pages. We invite readers and their students to create variations and extensions of them, for further research projects for undergraduates in topology, proofs, honors programs, or research experiences.

Our most challenging question concerns scarce limit topologies. The definition of a scarce limit topology – where limits are as rare as possible – arose through investigations of the discrete topology, where obstructions to limits (the existence of open sets) are as prevalent as possible. We saw that in every topology, a sequence that is eventually constant must converge to that constant. This led to Question 7, on scarce limit topologies. It is simple to state, but its solution is likely still open.

We hope the reader will enjoy pondering these ideas as much as we have. For further reflection, Figure 5 and Questions 17 and 18 furnish a few final thoughts that will help in answering Questions 10 and 16.

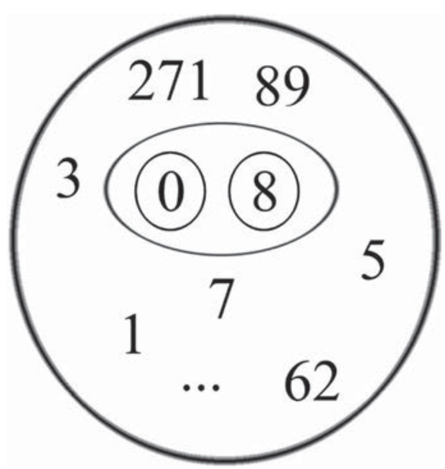


Figure 5. The topology on $\mathbb{N} \cup \{0\}$ whose open sets are $\emptyset$, $\mathbb{N} \cup \{0\}$, $\{0\}$, $\{8\}$, and $\{0, 8\}$. Does it have digressive convergence? Viewing this as an expanded beak topology will help in answering Question 16.

Question 17 ($\mathbb{N}$ versus $\mathbb{Q}$): Take the standard Euclidean topology on the set of rational numbers $\mathbb{Q}$, where the open sets are unions of open intervals in $\mathbb{Q}$. Is there any point with a minimal neighborhood? Since $\mathbb{Q}$ is countable, this topology will help in answering Question 10.

Question 18 (scarce limit sequences): A complementary concept to a scarce limit topology is a scarce limit sequence. Observe that if the only neighborhood of $p \in X$ is the whole set X, then p is a limit of every sequence in X. Such points p are the easiest to converge to; any sequence can do it! A sequence that can do no more than this we call a scarce limit sequence. That is, to say that a sequence (x_n) in X is a *scarce limit sequence* means that if (x_n) has limit p in some topology on X, then the only neighborhood of p is X. It is not hard to find scarce limit sequences on a finite set. Is there a scarce limit sequence on the infinite set $\mathbb{N}$? What about on $\mathbb{R}$?

APPENDIX

Definitions

(1) To say $S \subseteq \mathbb{R}$ is *open* means that for every $p \in S$, there is an open interval around p that is completely contained in S.
(2) A *topology* on a set X is a collection of subsets of X, called *open sets*, such that (1) X and the empty set are open, (2) arbitrary unions of open sets are open, and (3) finite intersections of open sets are open. The set X, together with its topology, is called a *topological space*.
(3) Let (x_k) be a sequence in $\mathbb{R}$, and let $p \in \mathbb{R}$. To say that *p is a limit of* (x_k) means that every open interval $(p - r, p + r)$, with $r > 0$, contains all but finitely many terms of (x_k).
(4) Let (x_k) be a sequence in a topological space X, and let $p \in X$. To say that *p is a limit of* (x_k) means that every open set containing p contains all but finitely many terms of (x_k).
(5) To say that a topological space X has the *scarce limit property* (or is a *scarce limit topology*) means that if a sequence in X has a limit, then the sequence is eventually constant.
(6) Let p be an element of a topological space X, and suppose that the intersection of all open sets containing p is open. We call this intersection the *minimal neighborhood* of p. To say that X has the *minimal neighborhood property* means that every point in X has a minimal neighborhood.
(7) Let (x_n) be a sequence with finite range in a topological space X. The *base set* of (x_n) is the set of all points in X that occur infinitely many times in the sequence. We say that two sequences are *equivalent* if they have the same base set. A sequence that converges to nothing in the base set but to at least one other element of X is said to have *digressive convergence*.

Topologies:

(1) Discrete topology on X: All subsets of X are open.
(2) Trivial topology on X: Only X and $\emptyset$ are open.
(3) Co-countable topology on X: The open sets are X, $\emptyset$, and every set whose complement is countable (that is, finite or countably infinite).
(4) Stalagmite topology on $\mathbb{N}$: The open sets are $\mathbb{N}$, $\emptyset$, and the sets $\mathbb{N}_k = \{1, 2, \ldots, k\}$.
(5) Stalactite topology on $\mathbb{N}$: The open sets are $\mathbb{N}$, $\emptyset$, and the sets $\mathbb{N}_k = \{k, k+1, k+2, \ldots\}$.
(6) Single 5 topology on $\mathbb{N}$: Only $\mathbb{N}$, $\emptyset$, and $\{5\}$ are open.
(7) Excluded 5 topology on $\mathbb{N}$: The open sets are $\mathbb{N}$ and all subsets that exclude 5.
(8) Candelabra topology on $\mathbb{Z}$: The open sets are arbitrary unions of the sets $\{-n, n\}$ and $\{0\}$.

(9) Winking owl topology on $X = \{0, -, V\}$: The open sets are X, $\emptyset$, $\{0\}$, $\{-\}$, and $\{0, -\}$.
(10) Expanded eye topology on $X = E_1 \cup E_2 \cup \{V\}$, where E_1 and E_2 are disjoint non-empty sets that do not contain V: The open sets are X, $\emptyset$, E_1, E_2, and $E_1 \cup E_2$.
(11) Parliament of owls topology on $E \cup V$, where E and V are disjoint non-empty sets: The open sets are X, $\emptyset$, and all subsets of E.

Questions:

(1) How can Euler's strategy of getting to the bottom of the question help you clarify other problems or ideas you are trying to understand?
(2) (a) Is the whole real line open? (b) Is the empty set open? (c) If we take any union of open sets in $\mathbb{R}$, is the result still open? (d) If we take the intersection of finitely many open sets in $\mathbb{R}$, is the result still open? (e) If we take any intersection of open sets in $\mathbb{R}$, is the result still open?
(3) How many topologies are there on a set of three points? What topologies are there on a set of two points? four points? How about one point? Can the set X be empty?
(4) (a) Can a topology on $\mathbb{R}$ have no open sets? If not, what are the fewest open sets it can have? (b) If we let every subset of $\mathbb{R}$ be open, does this make a topology?
(5) In the discrete topology on $\mathbb{R}$, does the sequence $(1/n)$ converge to 0 and only to 0, as usual? How does the topology help us to check?
(6) Is there any sequence that has a limit in the discrete topology? If so, what sequences do?
(7) Does every scarce limit topology on a set X contain the co-countable topology on X?
(8) In the stalagmite and stalactite topologies, what sequences converge to 4? For $p \in \mathbb{N}$, what sequences converge to p? How do the outcomes contrast?
(9) Does the single 5 topology have the minimal neighborhood property? What about the excluded 5 topology? What sequences have limit 5 in each of these topologies? Which have limit 7, or limit $p \neq 5$?
(10) Are there topologies on $\mathbb{N}$ for which no point has a minimal neighborhood, or where some, but not all points have minimal neighborhoods? What about topologies on $\mathbb{R}$?
(11) Does the sequence $(3, 4, 3, 4, \ldots)$ have a limit in the stalagmite or stalactite topologies, or in the single 5 or excluded 5 topologies? If so, what are the limits?
(12) Is there a simplest topology in which neither p nor q is a limit of $(p, q, p, q, \ldots)$ but something else is?
(13) How is the excluded 5 topology a generalization of the winking owl topology? Hint: picture the winking owl topology as an "excluded beak topology" where we think of the V as 5.
(14) In an expanded eye topology, what sequences have V as a unique limit but have no terms equal to V?
(15) How many equivalence classes of sequences does the winking owl topology have? Which of them have sequences with digressive convergence?
(16) What other topologies on a finite set have digressive convergence? What about topologies on an infinite set?

ORCID

Elizabeth R. Stephenson http://orcid.org/0000-0002-6862-208X

REFERENCES

[1] Arcavi, A. 2003. The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*. 52: 215–241.

[2] Burger, E. B. and M. Starbird. 2012. *The 5 Elements of Effective Thinking*. Princeton, NJ and Oxford, UK: Princeton University Press.

[3] Egenhofer, M. J. and R. D. Franzosa. 1991. Point-set topological spatial relations. *International Journal of Geographical Information System*. 5(2): 161–174.

[4] Euler, L. 1741. Solutio problematis ad geometriam situs pertinentis. *Euler Archive – All Works*. (53). https://scholarlycommons.pacific.edu/euler-works/53. Accessed 30 September 2020.

[5] Hege, H.-C. and K. Polthier, eds. 1997. *Visualization and Mathematics: Experiments, Simulations and Environments*. Heidelberg, Germany: Springer-Verlag.

[6] López, R. 2016. Cómo un topólogo clasifica las letras del alfabeto? *Miscelánea Matemática*. 61: 57–73.

[7] NOVA. 2006. NOVA – Transcripts – The Proof – PBS. PBS. September 2006. https://www.pbs.org/wgbh/nova/transcripts/2414proof.html. Accessed 18 Jan 2021.

[8] O'Connor, J. J. and E. F. Robertson. 2020. A History of Topology. University of Saint Andrews, st-andrews.ac.uk. https://mathshistory.st-andrews.ac.uk/HistTopics/Topology_in_mathematics/. Accessed 22 July 2020.

[9] Palais, R. S. 1999. The visualization of mathematics: Towards a mathematical exploratorium. *Notices of the AMS*. 46: 647–658.

[10] Sankaran, K. 2016. Beyond DIV, CURL, and GRAD: Modelling electromagnetic problems using algebraic topology. *Journal of Electromagnetic Waves and Applications*. 31(2): 121–149.

[11] Sethares, W. A. and R. Budney. 2014. Topology of musical data. *Journal of Mathematics and Music*. 8(1): 73–92.

[12] Shipman, B. A. 2013. Convergence and the Cauchy property of sequences in the setting of actual infinity. *PRIMUS*. 23(5): 441–458.

[13] Steen, L. A. and J. A. Seebach, Jr. 1978. *Counterexamples in Topology*, *Second Edition*. New York, NY: Springer Verlag.

[14] Tammadge, A. R. 1964. Stage A topology in the main school. *The Mathematical Gazette*. 48(366): 365–372.

BIOGRAPHICAL SKETCHES

Barbara A. Shipman enjoys helping people think about math in fun and creative ways. She likes to find and explain core concepts of how things work and loves to be out in nature or anywhere where there are curious students. She is a Distinguished Teaching Professor at the University of Texas at Arlington.

Elizabeth R. Stephenson stumbled across math unexpectedly and quickly grew quite fond of it. Now she enjoys thinking about topology and related concepts just as much as she loves painting, photography, knitting, reading, writing poetry, and drinking tea! She is a Ph.D. student at the Institute of Science and Technology Austria.

View publication stats