computational geometry & topology final project
survey: when does irreducibility imply minimality?

1 abstract

In this survey paper, we present results partially answering the following question posed in lecture 6: Can we understand the set of spaces (together with triangulations) for which irreducibility implies minimality? In particular, we classify closed surfaces according to this property. We present a small proof unifying the survey results.

2 definitions

These standard definitions are taken from [4] unless specified otherwise.

A **closed surface** is a collection M of triangles in some euclidean space such that
i. M satisfies the intersection condition
ii. M is connected
iii. for every vertex v of a triangle of M, the link of v is a simple closed polygon

the **intersection condition** states that two triangles either
i. are disjoint or
ii. have one vertex in common or
iii. have two vertices, and consequently the entire edge between them in common

A collection M of triangles satisfying the intersection condition is called **connected** if there is a path along the edges of the triangles from any vertex to any other vertex.

We define the **triangulation** of a topological space $\mathbb{X}$ to be a simplicial complex K, together with a homeomorphism $f : |K| \to \mathbb{X}$.

Let K be a triangulation of a 2-manifold $\mathbb{X}$. The edge contraction $f_{ab} : |K| \to |L|$ of edge ab preserves topological type iff Lka ∩ Lkb = Lkab, where Lkσ denotes the link of σ. K is **irreducible** if every edge contraction $f_{ab}, ab \in K$ changes topological type. [6]

Let B be a subset of K. We define the **closure** of B to be the set of all faces of simplices in B. $\overline{B} = \{\tau \in K | \tau \leq \sigma \in B\}$. The **star** of B is the set of all cofaces of simplices in B. St$B = \{\sigma \in K | \sigma \geq \tau \in B\}$. The **link** of B is the set of all faces of cofaces of simplices in B that are disjoint from simplices in B.
Lk$B = \{\overline{\mathrm{St}B} - \mathrm{St}\overline{B}\}$. [2]

We say that K is **minimum** if the cardinality of K is the smallest among all triangulations of $\mathbb{X}$.[6] We denote this property **minimality.** Note: sometimes in the literature, an irreducible triangulation is instead called minimal, and the corresponding property ''minimality''. Here, we use the wording we find most transparent.

The **connected sum** $M\#N$ of closed surfaces M and N is given by removing a triangle from both M and N, and then gluing the two surfaces together along the boundary of the missing triangle. The result is a closed surface, orientable iff M and N are both orientable. Furthermore, any two connected sums of M and N are equivalent.

3 classification theorem

We will make use of the following theorem:

Classification Theorem for Closed Surfaces Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori, or a sphere with a finite number of disjoint discs removed and with crosscaps glued in their place. The sphere and connected sums of tori are orientable surfaces, whereas surfaces with crosscaps are unorientable.

(this theorem is due to Möbius, von Dyck, Dehn, and Rado over a span of 30 years in the late 19th and early 20th century). [3]

reformulation Any closed surface is equivalent (homeomorphic) to exactly one of $\mathbb{S}$, $k\mathbb{P}$, or $k\mathbb{T}$, where $k\mathbb{X}$ denotes k connected sums of $\mathbb{X}$, and $\mathbb{S}, \mathbb{T}, \mathbb{P}$ refer to the sphere, torus, and projective plane respectively. [4]

For example, the Stanford bunny (which, unfortunately for the poor bunny, is hollow inside) is homeomorphic to a sphere, and may be triangulated with the boundary complex of the tetrahedron accordingly (See figure 1).

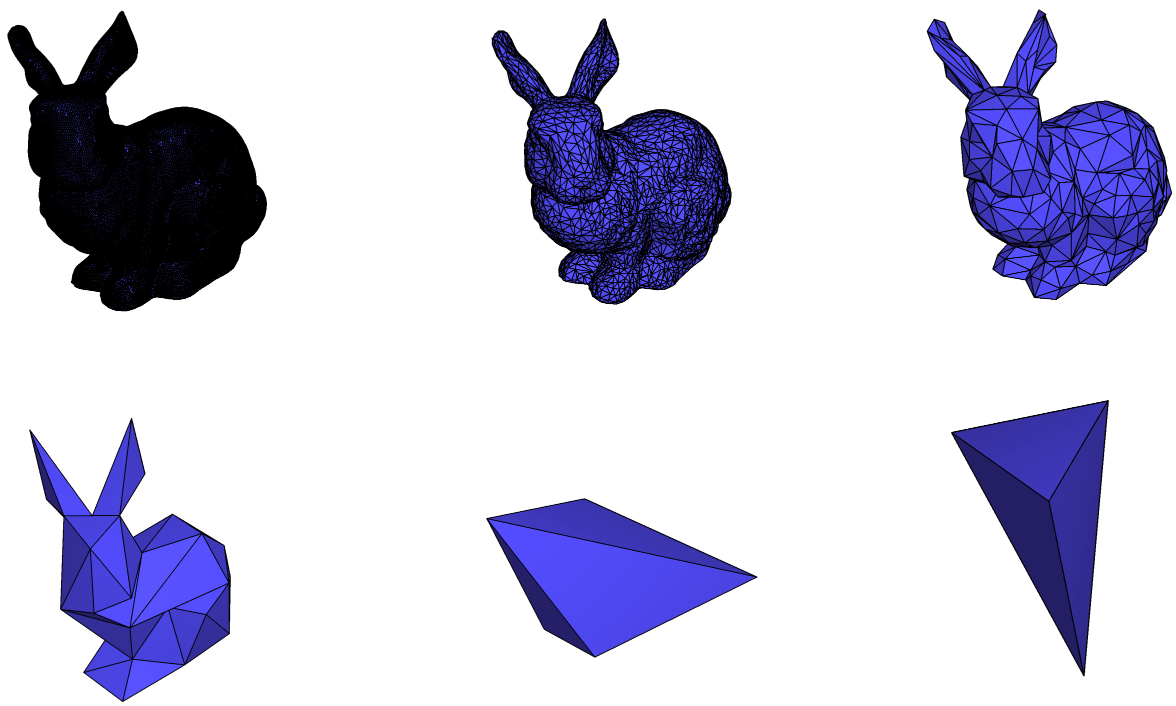

Figure 1: the stanford bunny turns into the tetrahedron boundary complex via a series of edge contractions (the top left appears black because the vertices and edges are so dense; it is, indeed, three-dimensional)

4 survey: irreducibility and minimality

The three theorems outlined in [6] together with results from [1] and [5] tell us the following:

(Steinitz 1934): K_4 is the only irreducible triangulation of $\mathbb{S}$. Thus, irreducibility indeed implies minimality for $\mathbb{S}$.

(Barnette 1982): The projective plane has 2 irreducible triangulations: one with 6 vertices, and the other with 7 [1]. Therefore irreducibility does not imply minimality for the projective plane.

(Lavrenchenko 1987): the largest irreducible triangulation of the torus has 10 vertices. However, there are smaller triangulations, such as the embedding of the complete graph with 7 vertices [5]. Therefore irreducibility does not imply minimality for the torus.

5 extension of the classification to connected sums

Thus, if a surface $\mathbb{X}$ is homeomorphic to $\mathbb{S}$, $\mathbb{T}$, or $\mathbb{P}$, we can immediately say whether irreducibility implies minimality for a triangulation of $\mathbb{X}$. (And the fact that homeomorphism is enough and that we do not need equality may be directly seen from the definition of a triangulation).

But what can we say about connected sums of $\mathbb{P}, \mathbb{T}$? It would be nice to combine the Classification Theorem for Closed Surfaces with these three theorems about triangulations. We now prove that the results extend easily to connected sums, as one would perhaps expect.

Theorem: A closed surface $\mathbb{X}$ homeomorphic to $k\mathbb{P}$ or $k\mathbb{T}$ has multiple irreducible triangulations with different numbers of vertices.

Throughout the proof we will denote $\mathbb{X}$ to be a closed surface homeomorphic to wlog $\mathbb{T}$, but our proof works for $\mathbb{P}$ as well. We also let K_1 and K_2 be triangulations of $\mathbb{X}$ which are both irreducible, but $\nu(K_1) < \nu(K_2)$, where we use $\nu(K)$ to denote the number of vertices of K.

We will also, through a slight abuse of notation, refer to the triangulation of a connected sum $(\mathbb{X}, K)\#(\mathbb{X}, K)$ of a surface $\mathbb{X}$ with triangulation K as $K\#K$. This is defined by the standard algorithm of removing one triangle from each copy of K, and then gluing the two surfaces together along the boundary of the empty triangles, yielding a surface which is again closed and connected. More details can be found in [4].

5.1 minimality in the connected sum

We want to show that, if a surface $\mathbb{X}$ has two triangulations which are both irreducible but have a different number of vertices, then taking k connected sums $k\mathbb{X}$ for each of the triangulations naturally yields two triangulations of $k\mathbb{X}$ which are both irreducible but have different numbers of vertices. Here we will show that we easily obtain two triangulations of $k\mathbb{X}$ with different numbers of vertices.

We know that $\nu(K_1) < \nu(K_2)$ by definition. We know also that $\nu(K_1\#K_1) = 2\nu(K_1) - 3$ (because we glue three vertices of each copy together), and also $\nu(K_2\#K_2) = 2\nu(K_2) - 3$. Therefore, $\nu(K_1\#K_1) < \nu(K_2\#K_2)$. Each time we take a connected sum, we lose three vertices from the total number of vertices in the summed spaces. Therefore in general, $\nu(nK) = n\nu(K) - 3(n-1)$, and $\nu(nK_1) < \nu(nK_2)$.

So we have obtained two triangulations $K_1\#K_1$ and $K_2\#K_2$ of $n\mathbb{X}$ which have different numbers of vertices. Next, we will prove that they are both still irreducible.

5.2 irreducibility in the connected sum

Here, we pick an arbitrary edge e from nK_1 or nK_2 (wlog let us say nK_1). We will show that we cannot contract it while retaining the topological type of the triangulation. We show this by attempting to contract it and then seeing that catastrophies arise as a result.

catastrophe one

Let $e = \{x, y\} \in$ the boundary of a removed triangle (from the construction of the connected sum). Then, contracting this edge contracts the entire boundary of the removed triangle to an edge, increasing the rank of the second homology group of the topological space by one. Therefore, this contraction is illegal. (One can also verify that it is illegal by checking the link condition: Lk$x \cap$ Lky does not equal Lkxy, because the former contains the third vertice of the removed triangle, whereas the latter does not). (Additionally, a third observation is that this contraction results in a stratified manifold rather than a manifold).

catastrophe two Let $e = x, y \notin$ the boundary of the removed triangle. (Note that it could connect to the removed triangle by one vertex).

It is useful to think of the edge contraction, a surjective simplicial map $f_{ab} : |K| \to |L|$, as a surjective vertex map

$$f(u) = \begin{cases} u, u \in vertices(K) - a, b \\ a, u \in a, b. \end{cases}$$

More details can be found in [2].

Suppose e connects to the removed triangle by one vertex, wlog x. The Link Condition was violated before we took the connected sum, and it is still violated, because while we may have modified the link of x, we did not modify Lk$x \cap$ Lky, nor Lkxy.

Even more clearly, if we suppose that e is not touching the removed triangle at all, we did not change the links of x, y, or xy by taking the connected sum, and so the Link Condition is still violated.

Therefore, we cannot contract any edges in the connected sum of nK_1 or nK_2, and both are irreducible.

6 discussion

In this paper, we highlighted results from Steinitz, Barnette, and Lavrenchenko about irreducible and minimal triangulations of the torus, sphere, and projective plane. We combined these with the classification of closed surfaces to characterize when irreducibility implies minimality for a triangulation of a closed surface. We found that irreducibility implies minimality exactly when the surface is homeomorphic to a sphere.

7 acknowledgements

This paper is the final project for the class ''Computational Geometry and Topology,'' taught at IST Austria by Herbert Edelsbrunner and Teresa Heiss in autumn 2019. The figures were generated with the CGAL 5.0 ''trianglulated surface mesh simplification'' algorithm based upon methods outlined in [2].

References

[1] David Barnette. Generating the triangulations of the projective plane. *Journal of Combinatorial Theory*, Series B 33:222--230, 1982.

[2] Tamal K. Dey, Herbert Edelsbrunner, Sumanta Guha, and Dmitry V. Nekhayev. Topology preserving edge contraction. *Pulications de L'Institut Mathématique*, 66(80):23--45, 1999.

[3] Zbigniew Fiedorowicz. Classification of surfaces. https://people.math.osu.edu/fiedorowicz.1/math655/classification.html. Accessed: 26 November 2019.

[4] Peter Giblin. *Graphs, Surfaces, and Homology*. Cambridge University Press, United Kingdom, 2010.

[5] M. Jungerman and G. Ringel. Minimal triangulations on orientable surfaces. *Acta Mathematica*, 145:121--154, 1980.

[6] Afra Zomorodian. Survey of results on minimal triangulations. *class project for CS 497, Edelsbrunner, UIUC*, 1998.

8 supplementary pretty pictures

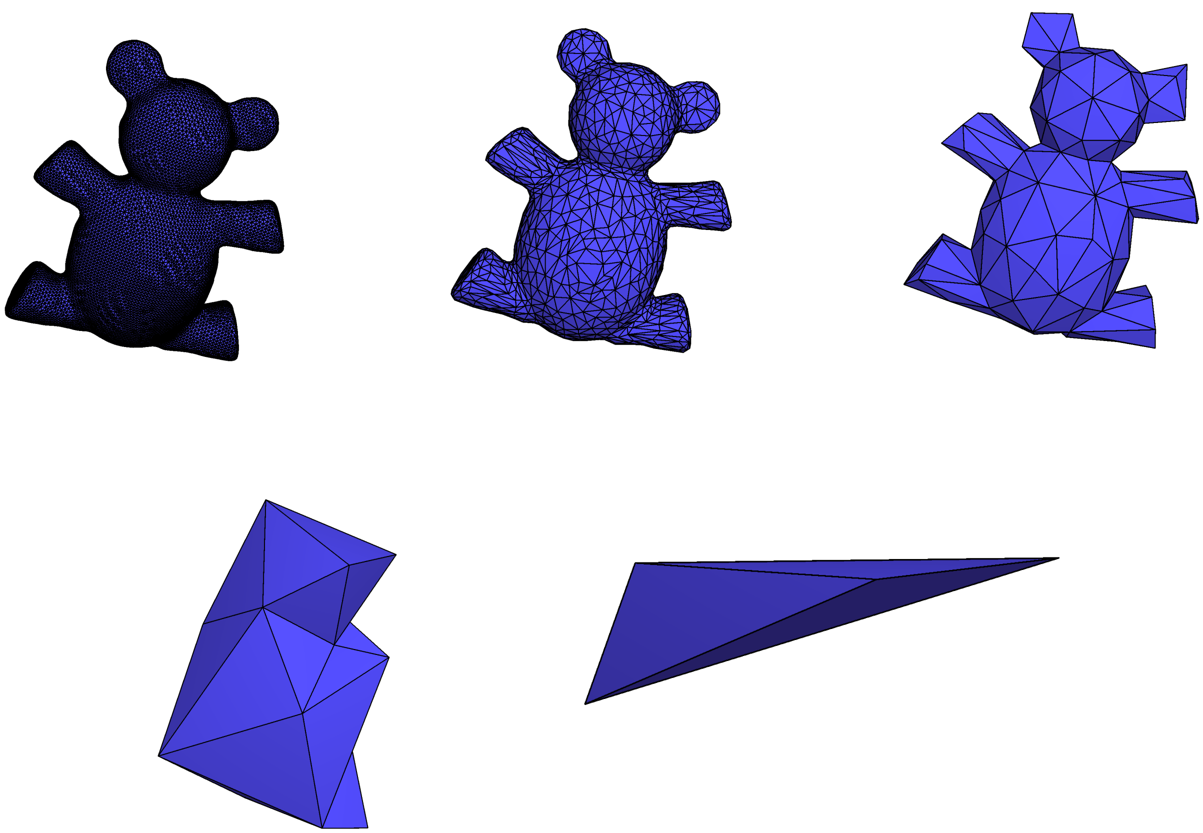

Figure 2: the bear contracts into a tetrahedron

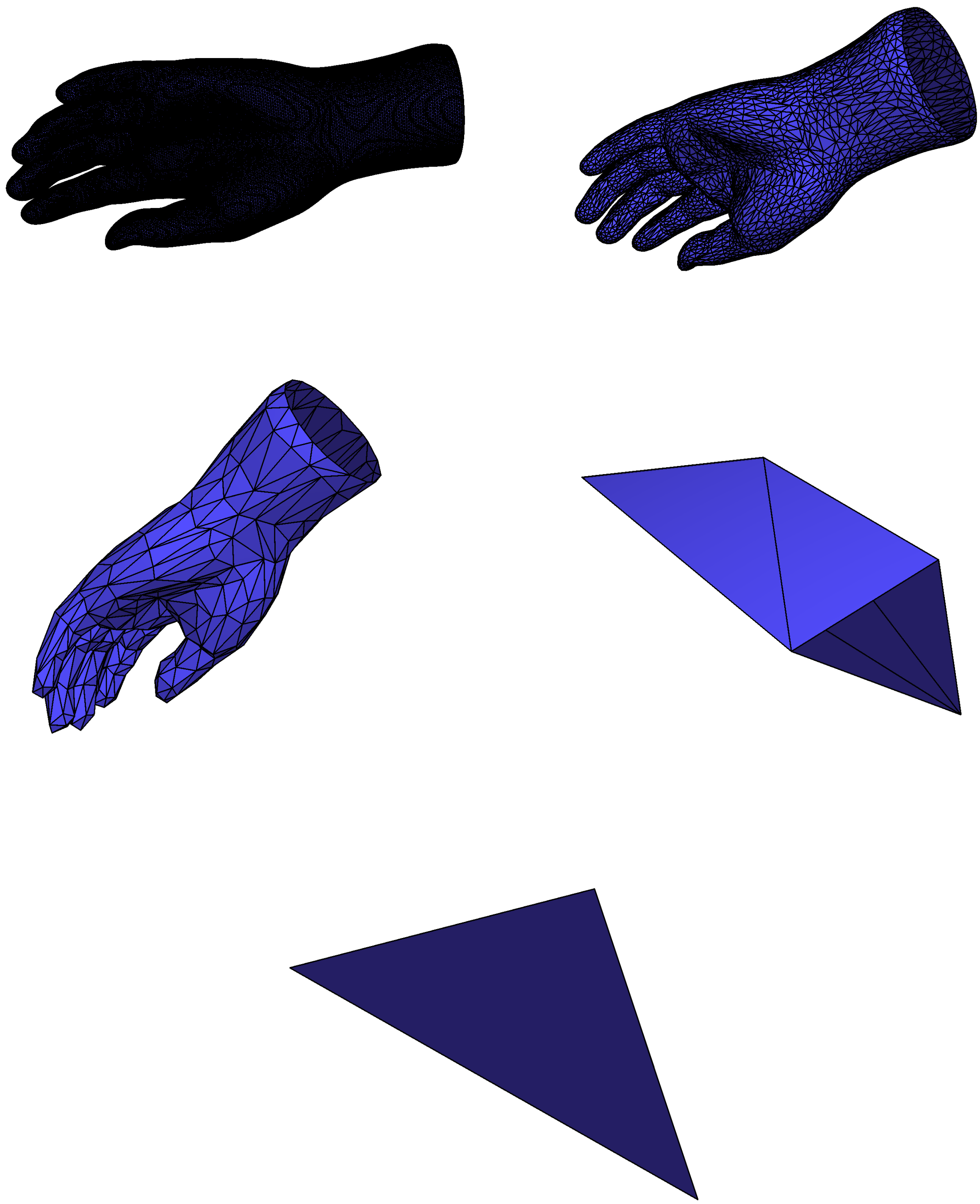

Figure 3: a glove contracts into a triangle (again, the top left appears black because of the density of edges: it is three-dimensional)

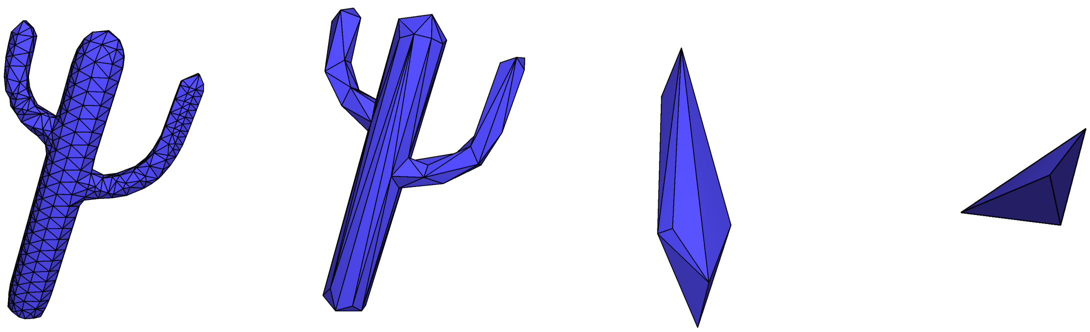

Figure 4: legend says that after a century without rain, the cacti in Texas turn into tetrahedra

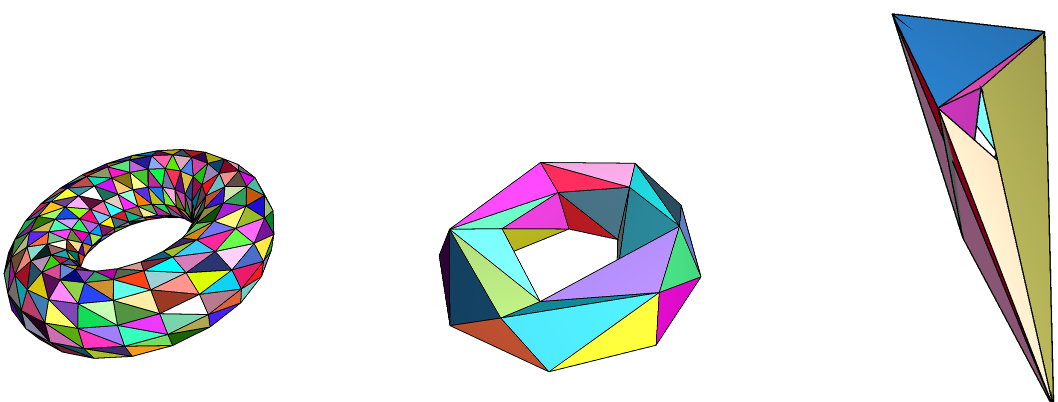

Figure 5: donut worry, I didn't forget to include a coffee cup ;)